Multiplication of Rank n Objects

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1 Origins

Suppose it is desired to multiply a rank 3 object with another rank 3 object and return a rank 3 object. Since any multiplication requires a contraction (a sum over indices of both objects), this is not possible – depending upon the indices chosen, only a rank 2 or rank 4 object can be obtained. So, multiplication must be modified. Let us define multiplication as follows

$$(ab)^i{}_{jk} \equiv a^i{}_{lm} f_j{}^{ln} b^m{}_{nk}$$

where a and b are rank three objects, the f is to be determined, and summation over repeated indices is implied. It is desirable to have the identity be δ_{abc} where

$$\delta_{abc} = \begin{cases} 1 & \text{if } a = b = c \\ 0 & \text{otherwise} \end{cases}$$

Then we require

$$\begin{aligned} (\delta a)^i{}_{jk} &= \delta^i{}_{lm} f_j{}^{ln} a^m{}_{nk} = a^i{}_{jk} \\ (a\delta)^i{}_{jk} &= a^i{}_{lm} f_j{}^{ln} \delta^m{}_{nk} = a^i{}_{jk} \end{aligned}$$

The first equation demands

$$\sum_n f_j{}^{in} a^i{}_{nk} = a^i{}_{jk}$$

$$\Rightarrow \quad f_j{}^{in} = \delta_j{}^n \eta^i \tag{1}$$

and the second equation requires

$$\sum_l a^i{}_{lk} f_j{}^{lk} = a^i{}_{jk}$$

$$\Rightarrow \quad f_j{}^{lk} = \delta_j{}^l \eta^k \tag{2}$$

Where

$$\eta^1 = \eta^2 = \eta^3 = 1$$

It is clear that both condition (1) and (2) can not be satisfied simultaneously.

2 Directed Multiplication

Since the two conditions can not be satisfied simultaneously, it is necessary to introduce the concept of directed multiplication. We define

$$
\begin{aligned}
(a \overset{\rightarrow}{\cdot} b)^i{}_{jk} &\equiv a^i{}_{mn}\, r_j{}^{mp}\, b^n{}_{pk} && \text{right directed multiplication} \\
(a \overset{\leftarrow}{\cdot} b)^i{}_{jk} &\equiv a^i{}_{mn}\, \ell_j{}^{mp}\, b^n{}_{pk} && \text{left directed multiplication}
\end{aligned}
$$

So that now

$$
\begin{aligned}
(\delta \overset{\rightarrow}{\cdot} a)^i{}_{jk} = a^i{}_{jk} \quad &\Rightarrow \quad r_j{}^{mk} = \delta_j{}^m \eta^k \\
(a \overset{\leftarrow}{\cdot} \delta)^i{}_{jk} = a^i{}_{jk} \quad &\Rightarrow \quad \ell_j{}^{ip} = \delta_j{}^p \eta^i
\end{aligned}
$$

Let us call the $r_i{}^{jk}$ and the $\ell_i{}^{jk}$ multiplication structure constants. The nonzero multiplication structure constants in $d = 3$ dimensions are

$$
\begin{aligned}
r_1{}^{11} = r_1{}^{12} = r_1{}^{13} = 1 \\
r_2{}^{21} = r_2{}^{22} = r_2{}^{23} = 1 \\
r_3{}^{31} = r_3{}^{32} = r_3{}^{33} = 1 \\
\ell_1{}^{11} = \ell_1{}^{21} = \ell_1{}^{31} = 1 \\
\ell_2{}^{12} = \ell_2{}^{22} = \ell_2{}^{32} = 1 \\
\ell_3{}^{13} = \ell_3{}^{23} = \ell_3{}^{33} = 1
\end{aligned}
$$

2.1 Properties of the Mult. Structure Constants

$$
r_i{}^{jk} = \ell_i{}^{kj}
$$

$$
\sum_i \sum_j \sum_k r_i{}^{jk} r_j{}^{ki} r_k{}^{ij} = d
$$

3 Rank 2

Left and right directed multiplication are equivalent for rank two objects:

$$
(A \overset{\rightarrow}{\cdot} B)^i{}_j \equiv A^i{}_k r^k{}_m B^m_j
$$

$$
(A \overset{\leftarrow}{\cdot} B)^i{}_j \equiv A^i{}_k \ell^k{}_m B^m_j
$$

The identity here is I, given by $I^i{}_j = \delta^i{}_j$ (the Kronecker delta symbol). Then

$$
\begin{aligned}
(I \overset{\rightarrow}{\cdot} A)^i{}_j &= A^i{}_j \\
\Rightarrow \quad \delta^i{}_k r^k{}_m A^m_j = r^i{}_m A^m_j &= A^i{}_j \\
\Rightarrow \quad r^i{}_m &= \delta^i{}_m
\end{aligned}
$$

$$
\begin{aligned}
(A \overset{\leftarrow}{\cdot} I)^i{}_j &= A^i{}_j \\
\Rightarrow \quad A^i{}_k \ell^k{}_m \delta^m_j = \ell^k{}_j A^i{}_k &= A^i{}_j \\
\Rightarrow \quad \ell^k{}_j &= \delta^k{}_j
\end{aligned}
$$

So that

$$(A \overset{\rightarrow}{\cdot} B)^i{}_j = A^i{}_k \delta^k{}_m B^m_j = A^i{}_m B^m_j$$

$$(A \overset{\leftarrow}{\cdot} B)^i{}_j = A^i{}_k \delta^k{}_m B^m_j = A^i{}_m B^m_j$$

Hence

$$(A \overset{\rightarrow}{\cdot} B)^i{}_j = (A \overset{\leftarrow}{\cdot} B)^i{}_j$$

4 Matrices

$$(R_i)_{jk} \equiv r_i{}^{jk}$$

$$(L_i)_{jk} \equiv \ell_i{}^{jk}$$

$$R_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad R_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad R_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$$L_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad L_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad L_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$L_i = R_i^T$$

$$R_i R_j = \ell_i{}^{jk} R_k$$

$$L_i L_j = r_k{}^{ji} L_k$$