

Multiplication of Rank n Objects

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1 Origins

Suppose it is desired to multiply a rank 3 object with another rank 3 object and return a rank 3 object. Since any multiplication requires a contraction (a sum over indices of both objects), this is not possible – depending upon the indices chosen, only a rank 2 or rank 4 object can be obtained. So, multiplication must be modified. Let us define multiplication as follows

$$(ab)^i_{jk} \equiv a^i_{lm} f_j^{ln} b^m_{nk}$$

where a and b are rank three objects, the f is to be determined, and summation over repeated indices is implied. It is desirable to have the identity be δ_{abc} where

$$\delta_{abc} = \begin{cases} 1 & \text{if } a = b = c \\ 0 & \text{otherwise} \end{cases}$$

Then we require

$$\begin{aligned} (\delta a)^i_{jk} &= \delta^i_{lm} f_j^{ln} a^m_{nk} = a^i_{jk} \\ (a\delta)^i_{jk} &= a^i_{lm} f_j^{ln} \delta^m_{nk} = a^i_{jk} \end{aligned}$$

The first equation demands

$$\begin{aligned} \sum_n f_j^{in} a^i_{nk} &= a^i_{jk} \\ \Rightarrow f_j^{in} &= \delta_j^n \eta^i \end{aligned} \tag{1}$$

and the second equation requires

$$\begin{aligned} \sum_l a^i_{lk} f_j^{lk} &= a^i_{jk} \\ \Rightarrow f_j^{lk} &= \delta_j^l \eta^k \end{aligned} \tag{2}$$

Where

$$\eta^1 = \eta^2 = \eta^3 = 1$$

It is clear that both condition (1) and (2) can not be satisfied simultaneously.

2 Directed Multiplication

Since the two conditions can not be satisfied simultaneously, it is necessary to introduce the concept of directed multiplication. We define

$$\begin{aligned} (a \xrightarrow{\cdot} b)^i_{jk} &\equiv a^i_{mn} r_j^{mp} b^n_{pk} && \text{right directed multiplication} \\ (a \xleftarrow{\cdot} b)^i_{jk} &\equiv a^i_{mn} \ell_j^{mp} b^n_{pk} && \text{left directed multiplication} \end{aligned}$$

So that now

$$\begin{aligned} (\delta \xrightarrow{\cdot} a)^i_{jk} = a^i_{jk} &\Rightarrow r_j^{mk} = \delta_j^m \eta^k \\ (\delta \xleftarrow{\cdot} a)^i_{jk} = a^i_{jk} &\Rightarrow \ell_j^{ip} = \delta_j^p \eta^i \end{aligned}$$

Let us call the r_i^{jk} and the ℓ_i^{jk} multiplication structure constants. The nonzero multiplication structure constants in $d = 3$ dimensions are

$$\begin{aligned} r_1^{11} = r_1^{12} = r_1^{13} &= 1 \\ r_2^{21} = r_2^{22} = r_2^{23} &= 1 \\ r_3^{31} = r_3^{32} = r_3^{33} &= 1 \\ \ell_1^{11} = \ell_1^{21} = \ell_1^{31} &= 1 \\ \ell_2^{12} = \ell_2^{22} = \ell_2^{32} &= 1 \\ \ell_3^{13} = \ell_3^{23} = \ell_3^{33} &= 1 \end{aligned}$$

2.1 Properties of the Mult. Structure Constants

$$\begin{aligned} r_i^{jk} &= \ell_i^{kj} \\ \sum_i \sum_j \sum_k r_i^{jk} r_j^{ki} r_k^{ij} &= d \end{aligned}$$

3 Rank 2

Left and right directed multiplication are equivalent for rank two objects:

$$\begin{aligned} (A \xrightarrow{\cdot} B)^i_j &\equiv A^i_k r^k_m B^m_j \\ (A \xleftarrow{\cdot} B)^i_j &\equiv A^i_k \ell^k_m B^m_j \end{aligned}$$

The identity here is I , given by $I^i_j = \delta^i_j$ (the Kronecker delta symbol). Then

$$\begin{aligned} & (I \vec{\cdot} A)^i_j = A^i_j \\ \Rightarrow \delta^i_k r^k_m A^m_j &= r^i_m A^m_j = A^i_j \\ \Rightarrow r^i_m &= \delta^i_m \end{aligned}$$

$$\begin{aligned} & (A \overleftarrow{\cdot} I)^i_j = A^i_j \\ \Rightarrow A^i_k \ell^k_m \delta^m_j &= \ell^k_j A^i_k = A^i_j \\ \Rightarrow \ell^k_j &= \delta^k_j \end{aligned}$$

So that

$$\begin{aligned} (A \vec{\cdot} B)^i_j &= A^i_k \delta^k_m B^m_j = A^i_m B^m_j \\ (A \overleftarrow{\cdot} B)^i_j &= A^i_k \delta^k_m B^m_j = A^i_m B^m_j \end{aligned}$$

Hence

$$(A \vec{\cdot} B)^i_j = (A \overleftarrow{\cdot} B)^i_j$$

4 Matrices

$$(R_i)_{jk} \equiv r_i^{jk}$$

$$(L_i)_{jk} \equiv \ell_i^{jk}$$

$$R_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad R_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad R_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$$L_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad L_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad L_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$L_i = R_i^T$$

$$R_i R_j = \ell_i^{jk} R_k$$

$$L_i L_j = r_k^{ji} L_k$$